Finsler manifolds with general symmetries

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Abstract

In this paper, we study generalized symmetric Finsler spaces. We first study symmetry preserving diffeomorphisms, then we show that the group of symmetry preserving diffeomorphisms is a transitive Lie transformation group. Finally we give some existence theorems.

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1 Introduction

Finsler manifold is a generalization of the Riemannian one, in the same as Riemann manifold is for the Euclidean. A metric depends on the point and the direction. A Finsler metric on a manifold is a family of Minkowski norms on tangent spaces.

Let (M, F) be a Finsler space, where F is positively homogeneous but not necessarily absolutely homogeneous. We introduce isometries of (M, F) which form a Lie transformation group on M as a result of [2] and moreover for any point $x \in M$, the isotropic subgroup $I_x(M, F)$ is a compact subgroup of $I(M, F)$, which can be used to study homogeneous and symmetric Finsler spaces [3, 7, 8].

Symmetric spaces have appeared to be very rich in content, stimulating the research in Lie groups, Mechanics, Physics, Gravity etc.

The definition of symmetric Finsler space is a naturall generalization of E. Cartan's definition of Riemannian symmetric spaces. We call a Finsler space (M, F) a symmetric Finsler space if for any point $p \in M$ there exists an involutive isometry s_p of (M, F) such that p is an isolated fixed point of s_p.

Affine and Riemannian $s-$manifold were first defined in [12] following the introduction of generalized Riemannian symmetric spaces in [13]. They form a more general class than the symmetric spaces. An isometry of (M, F) with an isolated fixed point $x \in M$ is called a symmetry of (M, F) at x. A family $\{s_x | x \in M\}$ of symmetries of a connected Finsler space (M, F) is called an $s-$structure of (M, F). In this paper we are concerned with properties of Finsler spaces admitting such an $s-$structure.

2 Preliminaries

Let M be an $n-$dimensional smooth manifold without boundary and TM denote its tangent bundle. A Finsler structure on M is a map $F : TM \longrightarrow [0, \infty)$ which has the following properties [1]:

1. F is smooth on $\widetilde{TM} := TM \backslash \{0\}$.
2. $F(x, \lambda y) = \lambda F(x, y)$, for any $x \in M, y \in T_xM$ and $\lambda > 0$.
3. F^2 is strongly convex, i.e.,

$$\mathrm{g}_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$$

   is positive definite for all $(x, y) \in \widetilde{TM}$.

Let $V = v^i \partial/\partial x^i$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric g_V and a linear connection ∇^V on the tangent bundle over $\mathcal{U}$ as following [1] :

$$g_V(X, Y) = X^i Y^j g_{ij}(x, v), \qquad \forall X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i},$$

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}(x, v) \frac{\partial}{\partial x^k}.$$

From the torsion freeness and $g-$compatibility of Chern connection we have

$$\nabla^V_X Y - \nabla^V_Y X = [X, Y],$$

$$X g_V(Y, Z) = g_V(\nabla^V_X Y, Z) + g_V(Y, \nabla^V_X Z) + 2C_V(\nabla^V_X V, Y, Z),$$

where C_V is the Cartan tensor defined by

$$C_V(X, Y, Z) = X^i Y^j Z^k C_{ijk}(x, v), \qquad C_{ijk}(x, v) = \frac{1}{4} \frac{\partial^3 F^2(x, v)}{\partial y^i \partial y^j \partial y^k},$$

Let $\gamma : [0, r] \longrightarrow M$ be a piecewise C^∞ curve. Its integral length is defined as

$$L(\gamma) = \int_0^r F(\gamma(t), \dot{\gamma}(t)) dt.$$

For $x_0, x_1 \in M$ denote by $\Gamma(x_0, x_1)$ the set of all piecewise C^∞ curve $\gamma : [0, r] \longrightarrow M$ such that $\gamma(0) = x_0$ and $\gamma(r) = x_1$. Define a map $d_F : M \times M \longrightarrow [0, \infty)$ by

$$d_F(x_0, x_1) = \inf_{\gamma \in \Gamma(x_0, x_1)} L(\gamma).$$

Of course we have $d_F(x_0, x_1) \geq 0$, where the equality holds if and only if $x_0 = x_1$; $d_F(x_0, x_2) \leq d_F(x_0, x_1) + d_F(x_1, x_2)$. In general, since F is only a positive homogeneous function, $d_F(x_0, x_1) \neq d_F(x_1, x_0)$, therefore (M, d_F) is only a non-reversible metric space.

Let (M, F) be a Finsler space, where F is positively homogeneous but not necessary absolutely homogeneous. As in the Riemannian case, we have two kinds of definitions of isometry on (M, F). On one hand, we can define an isometry to be a diffeomorphism of M onto itself which preserves the Finsler function. On the other hand, since on M we have the definition of distance function, we can define an isometry of (M, F) to be a mapping of M onto M which keeps the distance of each pair of points of M. The equivalence of these two definitions in the Finsler case is a result of S. Deng and Z. Hou [2]. They also prove that the group of isometries of a Finsler space is a Lie transformation group of the underlying manifold which can be used to study homogeneous Finsler spaces [3, 7, 8, 9, 10, 11].

3 Generalized symmetric Finsler spaces

Affine and Riemmannian $s-$manifolds were first defined in [12] following the introduction of generalized Riemannian spaces in [13]. They form a more general class than the symmetric spaces of E. Cartan [5, 6]. The definition of generalized symmetric Finsler space is a natural generalization of definition of generalized Riemannian symmetric spaces [4].

Definition 3.1 *Let (M, F) be a connected Finsler space. An isometry on (M, F) with an isolated fixed point x will be called a symmetry at x, and will usually be written as s_x.*

Definition 3.2 *A family $\{s_x | x \in M\}$ of symmetries on a connected Finsler manifold (M, F) is called an $s-$structure on (M, F)*

An $s-$structure $\{s_x | x \in M\}$ is called of order k $(k \geq 2)$ if $(s_x)^k = id$ for all $x \in M$ and k is the least integer of this property. Obviously a Finsler space is symmetric if and only if it admits an $s-$structure of order 2. An $s-$structure $\{s_x | x \in M\}$ on (M, F) is called regular if for every pair of points $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y).$$

Definition 3.3 *A generalized symmetric Finsler space is a connected Finsler manifold (M, F) admitting a regular $s-$structure and a Finsler space (M, F) is said to be $k-$symmetric $(k \geq 2)$ if it admits a regular $s-$structure of order k.*

Given an $s-$structure $\{s_x|x \in M\}$ on (M, F) we shall always denote by S the tensor field of type $(1, 1)$ defined by $S_x = (s_{x*})_*$ for all $x \in M$. Suppose there exists a nonzero vector $X \in T_xM$ such that $S_xX = X$. Since s_x is isometry, $s_x(\exp_x(tX))$, $|t| < \epsilon$ is a geodesic. Now $\exp_x(tX)$ and $s_x(\exp_x(tX))$ are two geodesics through x with the same initial vector X. Therefore, for any $|t| < \epsilon$ we have

$$s_x(\exp_x(tX)) = \exp_x(tX).$$

But this contradicts to assumption that x is an isolated fixed point of s_x. Therefore S_x has no non-zero invariant vector.

Theorem 3.1 *Let (M, F) be a generalized symmetric Finsler space. Then the tensor field S is invariant with respect to all symmetries s_x, i.e.*

$$s_{x*}(S) = S, \quad x \in M.$$

Proof: Let $\{s_x|x \in M\}$ be the $s-$structure of (M, F). From $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$ we obtain $(s_{x*})_y \circ S_y = S_z \circ (s_{x*})_y$ at the point $y \in M$. Hence $s_{x*} \circ S = S \circ s_{x*}$ holds on the tangent bundle TM, and this is the invariance of S with respect to symmetries.$\Box$

Theorem 3.2 *Let (M, F) be a Finsler space and $\{s_x\}$ a regular $s-$structure on M. Then there is a unique connection $\widetilde{\nabla}$ on M such that*

(i) $\widetilde{\nabla}$ *is invariant under all s_x*

(ii) $\widetilde{\nabla}S = 0$

If the Finsler space (M, F) is of Berwald type, then $\widetilde{\nabla}$ is given by the formula

$$\widetilde{\nabla}_X Y = \nabla_X Y - (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y)$$

Proof: The proof is similar to the Riemannina case [6].

Definition 3.4 *Let (M, F) be a generalized symmetric Finsler space, and let $\{s_x\}$ be the regular $s-$structure of (M, F). Then a diffeomorphism $\phi : M \longrightarrow M$ is called symmetry preserving if $\phi(s_x(y)) = s_{\phi(x)}\phi(y)$ for all $x, y \in M$.*

Obviously, all symmetries s_x are symmetry preserving due to $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$. We denote the group of symmetry preserving diffeomorphism by $Aut(\{s_x\})$. Let us denote by $A(M)$ the Lie group of all affine transformations of M with respect to the connection $\widetilde{\nabla}$. Each symmetry preserving diffeomorphism is an affine transformation of $(M, \widetilde{\nabla})$, i.e.

$$Aut(M, \{s_x\}) \subset A(M).$$

Lemma 3.1 *An affine transformation $\phi \in A(M)$ is symmetry preserving if and only if it preserves the tensor field S. Consequently, $Aut(\{s_x\})$ is a closed subgroup of $A(M)$ and hence a Lie transformation group of M.*

Proof: Let $\phi \in A(M)$ be symmetry preserving transformation then for each $x \in M$, maps $\phi \circ s_x$, $s_{\phi(x)} \circ \phi$ coincide, so $(\phi \circ s_x)_{*x} = (s_{\phi(x)} \circ \phi)_{*x}$. Then ϕ preserves the tensor field S. On the other hand if $\phi \in A(M)$ preserves the tensor field S then for each $x \in M$, $(\phi \circ s_x)_{*x} = (s_{\phi(x)} \circ \phi)_{*x}$. Because $\phi \circ s_x$ and $s_{\phi(x)} \circ \phi$ are affine transformations, so $\phi \circ s_x = s_{\phi(x)} \circ \phi$ that is ϕ is symmetry preserving map.$\square$

In the following we show that the group $Aut(\{s_x\})$ of all symmetry preserving diffeomorphisms of (M, F) is a transitive Lie transformation group.

Theorem 3.3 *The Lie transformation group $Aut(\{s_x\})$ act transitively on M.*

Proof: Let $K \subset Aut(\{s_x\})$ be the transformation group of M generated algebraicaly by all the symmetries s_x, $x \in M$. Choose an origin $o \in M$. Let $K(o)$ be the orbit of o with respect to K. Consider the map $f(x) = s_x(p)$ where $p \in K(o)$ and $x \in M$. Clearly $f(p) = p$. For $v \in T_pM$ we have $f_{*p}(v) = (I_p - S_p)v$. Hence $f_{*p} = (I_p - S_p)$ is a non-singular transformation and f maps a neighborhood U of p diffeomorphically onto a neighborhood V of p. We get $V \subset K(o)$ and the orbit $K(o)$ is open. The union of all other orbits of K must be also open and hence $K(o)$ is closed. Consequently $K(o) = M$. $\square$

Let V be a finite dimensional vector space and $T : V \longrightarrow V$ an endomorphism. Then there is a unique decomposition $V = V_{0T} + V_{1T}$ of V into $T-$invariant subspaces such that the restriction of T to V_{0T} is nilpotent and the restriction of T to V_{1T} is an automorphism.

Definition 3.5 *A regular homogeneous $s-$manifold is a triplet (G, H, σ), where G is a connected Lie group, H its closed subgroup and σ an automorphism of G such that*

(i) *$G_\sigma^\circ \subset H \subset G_\sigma$ where G_σ is the subgroup consisting of the fixed points of σ in G and G_σ° denotes the identity component of G_σ.*

(ii) *If T denotes the linear endomorphism $Id - \sigma_*$, then $\mathfrak{g}_{0T} = \mathfrak{h}$.*

Theorem 3.4 *Let (G, H, σ) be a regular homogeneous $s-$manifold with the $G-$invariant Finsler metric on G/H such that the transformation s of G/H determined by σ i.e. $s \circ \pi = \pi \circ \sigma$ is metric preserving at the origin eH of G/H. Then G/H is a generalized symmetric Finsler space and the symmetries s_x are given by*

$$s_{\pi(g)} = g \circ s \circ g^{-1}, \qquad g \in G, x = \pi(g).$$

Proof: Choose $g \in G$ and $x \in M$ then $x = \pi(g')$ for some $g' \in G$. Now,

$$\begin{aligned}
(s \circ g \circ s^{-1})(x) &= (s \circ g \circ s^{-1} \circ \pi)(g') \\
&= (s \circ g \circ \pi)(\sigma^{-1}(g')) \\
&= (s \circ \pi)(g\sigma^{-1}(g')) \\
&= (\pi \circ \sigma)(g\sigma^{-1}(g')) \\
&= \pi(\sigma(g)g') = \sigma(g)[\pi(g')] = \sigma(g)(x).
\end{aligned}$$

Hence we get

$$s \circ g \circ s^{-1} = \sigma(g) \qquad g \in G \tag{1}$$

So for $h \in H$ we obtain $s \circ h \circ s^{-1} = h$ and hence $h \circ s \circ h^{-1} = s$. Consequently the transformation $g \circ s \circ g^{-1}$ always depends only on $\pi(g)$ and

$$s_{\pi(g)} = g \circ s \circ g^{-1} \qquad g \in G$$

defines a family $\{s_x | x \in M\}$ of diffeomorphisms of M. We can also easily that $(x, y) \longrightarrow s_x(y)$ is differentiable. Further for $x \in M$, $x = \pi(g)$ we have $x = g(o)$ and hence

$$s_x(x) = (g \circ s \circ g^{-1})(x) = x,$$

because $s(o) = o$.
Now for $x, y \in M$ put $s_x = g \circ s \circ g^{-1}$, $s_y = g' \circ s \circ (g')^{-1}$, where $x = g(o)$ and $y = g'(o)$. Then

$$(g \circ s \circ g^{-1} \circ g' \circ s^{-1})(o) = s_x(g'(o)) = s_x(y),$$

on the other hand, (1) yields $g \circ s \circ g^{-1} \circ g' \circ s^{-1} = g\sigma(g^{-1}g')$. Thus, the map $g \circ s \circ g^{-1} \circ g' \circ s^{-1}$ coincides with the action of an element $g'' \in G$, $g''(o) = s_x(y)$. Now

$$\begin{aligned} s_x \circ s_y &= g \circ s \circ g^{-1} \circ g' \circ s \circ (g')^{-1} \\ &= g'' \circ s \circ (g'')^{-1} \circ g \circ s \circ g^{-1} \\ &= s_{s_x(y)} \circ s_x. \end{aligned}$$

It remains to prove that s_{x*} has no fixed vector except the null vector. If we identify $\mathfrak{g}$ with T_eG, then the projection $\pi_{*e} : T_eG \longrightarrow T_oM$ induces an isomorphism of $\mathfrak{g}_{1T}$ onto T_oM. From the relation $\pi_* \circ \sigma_* = s_* \circ \pi_*$ we can see that $\pi_* \circ T = (I_o - s_{*o}) \circ \pi_*$. Because T is an automorphism on $\mathfrak{g}_{1T}$, $I_o - s_{*o}$ is an automorphism of T_oM. From

$$s_{\pi(g)} = g \circ s \circ g^{-1}, \qquad g \in G, x = \pi(g),$$

we obtain easily that $I_p - S_p$ is an automorphism of T_pM for each $p \in M$. Thus $\{s_x | x \in M\}$ is a regular $s-$structure on (M, F).$\square$

Let $k \geq 2$ be an integer. A generalized symmetric Finsler space (M, F) is said to have order k if $(s_x)^k = id$ for all $x \in M$, and k is the least integer with this property.

Definition 3.6 *A regular homogeneous $s-$manifold (G, H, σ) is said to have order k if $\sigma^k = id$, and k is the least integer with this property.*

Theorem 3.5 *Let G be a connected Lie group, H its closed subgroup and σ an automorphism of G such that*

(i) $G_\sigma^\circ \subset H \subset G_\sigma$

(ii) $\sigma^k = id$ *(k being the minimum number with this property)*

Then (G, H, σ) is a regular homogeneous $s-$manifold of order k.

Proof: Let σ_* be the induced automorphism of the Lie algebra $\mathfrak{g}$ of G, and put $T = id - \sigma_*$. We have to show that $\mathfrak{g}_{0T} = \mathfrak{h}$. Here obviously $\mathfrak{h} = ker\ (T)$ and hence $\mathfrak{h} \subset \mathfrak{g}_{0T}$. Suppose now that there is $X \in \mathfrak{g}_{0T}$ such that $X \in (\mathfrak{g}_{0T} - \mathfrak{h})$. We can assume, without loss of generality, $TX \neq 0$, $T^2X = 0$. Then we get $\sigma_*(X) = X - Z$, where $\sigma_*(Z) = Z$. Hence we obtain by the induction $(\sigma_*)^2X = X - 2Z$, ...,$(\sigma_*)^kX = X - kZ$. Because $(\sigma_*)^kX = X$, we get $Z = 0$, a contradiction. This complete the proof. $\square$

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